RESEARCH IN LOGISTICS & PRODUCTION

# MINIMIZATION OF NUMBER OF BUSES IN THE SCHOOL BUS ROUTING PROBLEM

Kazimierz Worwa

Faculty of Cybernetics, Military Technical University, Warsaw, Poland,
Email: kazimierz.worwa@wat.edu.pl

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**Abstract:** In this paper a formal presentation and description of a method of solving the problem of both determining the set of bus stops and assignment of students that are authorized to transport to these stops is investigated. This problem can be treated as a subproblem of the bus school routing problem (SBRP). Although the problems of the SBRP class are one of the earliest logistics problems solved using methods of operations research, they remain valid and are the subject of research, as evidenced by numerous contemporary publications. Unfortunetly, in most of the problems of SBRP class described in the literature the problem of determining the bus stops network and allocation of students to the particular stops is very often ignored. Based on the assumption that a small number of bus stops, from which the students are taken or to which they are set down, makes carrying out of school transport process easier, a problem of minimizing the number of active bus stops was considered. The main result of this paper is proposition of a greedy algorithm to solving the problem of determining the minimum set of school bus stops. To illustrate functioning the proposed algorithm a simple numerical example has been presented.

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## 1. INTRODUCTION

One of the areas of modern logistics, understood as a process of planning, implementing and controlling the efficient and cost-effective flow of raw materials, finished products or people is transport logistics, dealing with the planning and optimization of the movement of materials, products and people. Typical transport logistics problems include the task of determining optimal routes for the group (fleet) vehicles, called problem routing vehicle (Vehicle Routing Problem, VRP). The essence of the routing problem is to determine the optimal routes for a number of means of transport, whose task is to handle the needs of customers located in different locations, fulfilling a number of limitations. In practice, as an optimization criterion is often assumed the total cost of transport, expressed in terms of distance, time or cost. Vehicle routing problem is one of the basic problems of the operational management of transport fleet, acting a classic example of a complex logistical problem solved based on the use of selected research methods of system analysis, with particular emphasis on methods of mathematical modeling and optimization. Due to the fact that the costs of transportation and distribution of goods and people are very important elements of operating costs of all economic and social organizations, the routing problems remain valid and up to date over the past few years, and different variants of these problems are commonly found in contemporary logistics.

An important recent example of the routing problem is the problem of determining the optimal route for a fleet of school buses (School Bus Routing Problem, SBRP). While the typical vehicle routing problems mainly relate to transport of goods, SBRP problems are closely related to the transport of people, e.g. students. Descriptions of practical problems SBRP found in the literature differ in details the assumptions, limitations and additional conditions that must be met obtained from their optimal or suboptimal buses route. Because of the large number and complexity of these limitations, school bus routing problems are often even more complex than the typical routing vehicle problems. A typical school bus routing problem (SBRP) can be characterized as follows: a group of spatially distributed students must provide school transport from places of residence to school or from school to places of residence. The problem is to determine a set of bus routes, selected from the available fleet of vehicles (which may include vehicles of different types, in particular with different numbers of seats), providing transportation for all eligible students, while providing additional conditions, such as minimizing transportation costs, minimizing the duration of the process of students transportation, minimizing the number of needed buses, etc. The school bus routing problem (SBRP) consists of two main component subproblems:

1. identification of a set of stops where buses stop and assign students to those stops, including restrictions on the distance between the places of students residence and bus stops;

2. the determination of bus routes and timetables for individual stops.

Despite that the SBRP problems were among the first logistical problems solved using the methods of operational research, they are still very valid and remain the subject of study and research, as evidenced by numerous contemporary publications, presenting new methods of their formal formulation and solving. In (Spasovic et al., 2001, Spada et al., 2005, Worwa, 2014) the problem of routing a fleet of school buses is analyzed, through its formal formulation and solution in relation to the case study.

The main aim of the considerations contained in this article is a formal presentation and description of a method to solve the first of the aforementioned subproblems of bus school routing problem (SBRP), ie. the problem of determining the network of bus stops and assigning students to particular stops, including restrictions on the distance between the places of residence of students and bus stops. Considerations set out later in this article are organized as follows: parts 2 and 3 are respectively contain general and mathematical description of the problem in question. Part 4 presents a numerical example illustrating the formulation of the problem and the method of its solution, while part 5 contains a summary of the considerations.

## 2. GENERAL DESCRIPTION OF THE PROBLEM OF DETERMINING A SET OF SCHOOL BUS STOPS

Determination of the bus stops location needs to take into account places of students residence that are eligible to transport. The basic condition that must be taken into account when determining the location of the stops is to ensure that the distance from the place of residence of each student and its assigned bus stop does not exceed a certain value, e.g. 1 km or 10 minutes walking distance. In this aspect, it is also clear requirement that the total number of students assigned to specific bus stop for each bus route does not exceed the permissible number of seats on the bus. It is worth mentioning that in most of the problems of SBRP class described in the literature the problem of determining the location of bus stops and assigning students to particular stops is ignored (it is assumed that the location of bus stops and assign to them the students are given). This problem is taken in the limited works only, such as in (Park and Kim, 2010), wherein for its solutions are usually used heuristic classified into two groups of methods: LAR (*Location-Allocation-Routing*) or ARL (*Allocation -Routing-Location*).

Both in practice and in the literature a number of restrictions, conditions and requirements are taken into account when formulating and solving problems SBRP class. Among the most commonly groups restrictions that are taken into account can be mentioned (Spada et al., 2005): a limit on the maximum number of seats on the bus, a limit on the maximum time limit for driving a bus, a limit on the maximum allowable time of each student on the bus (travel time to the school), a limit on the maximum permissible distance that the student can cover on foot on the way to set for his bus stop, restrictions on the "time windows" trips of buses on the route and arrival to school, restrictions on the number of students assigned to individual bus stop, restrictions on the minimum allowable number of students for which a route is created, etc. As a criterion functions (quality indicators solution) in SBRP problems are often used overall cost of transporting students or duration of transport.

## 3. MATHEMATICAL FORMULATION OF THE PROBLEM TO DETERMINE A SET OF SCHOOL BUS STOPS WITH A MINIMUM CARDINALITY

According to previous remarks considerations contained in this paper do not cover all aspects of a typical SBRP problem and only applies to the problem of determining the set of bus stops where buses stop and assigning students to particular stops, including restrictions on the distance between the places of residence of students and bus stops.

Let $\overline{\boldsymbol{P}}$ denote the set of numbers of potential bus stops, $\overline{\boldsymbol{P}} = \{1,2,3,...,\overline{p},...,\overline{P}\}$, where potential bus stop means a place on one of the roads on which school buses can move, allowing the bus to stop, taking into account the applicable rules of the road. In further discussion, it is assumed that a set of bus stops $\boldsymbol{P}$, $\boldsymbol{P} = \{1,2,3,...,p,...,P\}$, to which will be assigned individual, authorized to transport, students, is a subset of the set $\overline{\boldsymbol{P}}$, i.e. $\boldsymbol{P} \subseteq \overline{\boldsymbol{P}}$. Let $\boldsymbol{U}$, $\boldsymbol{U} = \{1,2,3,...,u,...,U\}$, denote a set of all numbers of students authorized to transport. Let $D = [d_{\overline{p}u}]_{\overline{P} \times U}$ mean matrix of distances from individual places of students residence, authorized to transport, and various potential bus stops, where element $d_{\overline{p}u} \geq 0$ determines the distance from bus stop number $\overline{p}$ to a place of residence-of a student number $u$, $\overline{p} \in \overline{\boldsymbol{P}}$, $u \in \boldsymbol{U}$, where the distance may be expressed in units of distance or time.

Let $\overline{X} = [\overline{x}_{\overline{p}u}]_{\overline{P} \times U}$ mean a matrix of elements which determine the potential of individual assignments of students, authorized to transport, to potential bus stops, where the element $\overline{x}_{\overline{p}u} \geq 0$, $\overline{p} \in \overline{\boldsymbol{P}}$, $u \in \boldsymbol{U}$, is defined as follows:

$$\overline{x}_{\overline{p}u} = \begin{cases} 1 & \text{if } \; d_{\overline{p}u} \leq d_{max} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where the value $d_{max}$ is the maximum allowable distance from the place of residence of the student and a bus stop. The matrix $\overline{X}$ elements have the following properties:

- $\sum_{u=1}^{U} \overline{x}_{\overline{p}u} \geq 0$, $\overline{p} \in \overline{\boldsymbol{P}}$, i.e. that every potential bus stop can be assigned to one or more students; it is also possible situation that potential bus stop remains unused, i.e. $\sum_{u=1}^{U} \overline{x}_{\overline{p}u} = 0$, which means that he was not assigned to any student;
- $\sum_{\overline{p}=1}^{\overline{P}} \overline{x}_{\overline{p}u} \geq 1$, $u \in \boldsymbol{U}$, i.e. that everyone, authorized to transport, student must be assigned to at least one potential stop; if the place of residence of a particular student is less than $d_{max}$ from more than one stop than $\sum_{\overline{p}=1}^{\overline{P}} \overline{x}_{\overline{p}u} > 1$.

In further considerations bus stop $\overline{p} \in \overline{\boldsymbol{P}}$ will be called:

- an active bus stop, if it has been assigned to at least one student, i.e. when there is $\sum_{u=1}^{U} \overline{x}_{\overline{p}u} > 0$,
- a passive bus stop, if no student is assigned to him, i.e. when there is $\sum_{u=1}^{U} \overline{x}_{\overline{p}u} = 0$.

The matrix $\overline{X}$ that describes authorized to transport students to potential bus stops, elements of which satisfy the condition (1), provides the basis for defining a set $\boldsymbol{X}$ of matrix $X = [x_{\overline{p}u}]_{\overline{P} \times U}$ which elements are defined as follows:

$$x_{\overline{p}u} = \begin{cases} 1 & if \;\; \overline{x}_{\overline{p}u} = 1 \\ 0 & if \;\; \overline{x}_{\overline{p}u} = 0 \;\; or \;\; \overline{x}_{\overline{p}u} = 1. \end{cases} \tag{2}$$

Matrices $X \in \boldsymbol{X}$ and the matrix $\overline{X}$ have the same dimensions and zero-one elements, wherein according to the condition (1), the element $x_{\overline{p}u}$ of matrix $X$ may be equal to *1* only if the corresponding element $\overline{x}_{\overline{p}u}$ of the matrix $\overline{X}$ is also equal to *1*. It is however possible situation that, although the element $\overline{x}_{\overline{p}u}$ of the matrix $\overline{X}$ is equal to *1*, that element $x_{\overline{p}u}$ of the matrix $X$ is equal to *0*. In addition, if the element $\overline{x}_{\overline{p}u}$ of the matrix $\overline{X}$ is equal to *0*, i.e. that to the bus stop number $\overline{p}$ can not be assigned student number *u* that element $x_{\overline{p}u}$ of the matrix *X* must also be equal to $0$ .

In further consideration the elements of the matrix $X \in \boldsymbol{X}$ will define the possible assignment of students to the respective bus stops. In the problem of determining the set of school bus stops that is investigated in this paper matrix $X \in \boldsymbol{X}$ will be treated as a decision matrix and the set of ***X*** will constitute a set of acceptable arrays (solutions).

The matrix $X \in \boldsymbol{X}$ defines the permissible assignment authorized to transport students to potential bus stops if the elements have the following properties:

- $x_{\overline{p}u} = \begin{cases} 1 & \text{if to the bus stop number } \overline{p} \text{ is assigned the student number } u, \\ 0 & \text{otherwise;} \end{cases}$
- $\sum_{u=1}^{U} x_{\overline{p}u} \geq 0$, $\overline{p} \in \overline{\boldsymbol{P}}$, i.e. that to every potential bus stop $\overline{p} \in \overline{\boldsymbol{P}}$ can be assigned one or more students (in this case the stop becomes the active stop); it is also possible situation that potential bus stop $\overline{p} \in \overline{\boldsymbol{P}}$ remains unassigned (passive), i.e. $\sum_{u=1}^{U} x_{\overline{p}u} = 0$ which means that there is no student assigned to it;
- $\sum_{\overline{p}=1}^{\overline{P}} x_{\overline{p}u} = 1$, $u \in \boldsymbol{U}$, i.e. that everyone authorized to transport student must be assigned to exactly one stop.

Based on the assumption that a small number of bus stops, of which the students are taken or to which they are set down makes it easier carrying out of school transport, later in this paper a problem of minimizing the number of active bus stops will be considered.

Each matrix $X \in \boldsymbol{X}$ defining assignment of students to particular bus stops clearly defines a set of numbers of active bus stops, i.e. those bus stops which will be used by authorized to transport students. To identify the active bus stops vector $s(X) = \left(s_1(X), s_2(X), \ldots, s_{\overline{p}}(X), \ldots, s_{\overline{P}}(X)\right)$ will be used such that

$$s_{\overline{p}}(X) = \begin{cases} 1 & \sum_{u=1}^{U} x_{\overline{p}u} > 0, \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

wherein $s_{\bar{p}}(X) \in \bar{\boldsymbol{P}}$. Then the set of active bus stops $\boldsymbol{P} \subseteq \bar{\boldsymbol{P}}$ can be obtained by assigning to it these numbers from the set $\bar{\boldsymbol{P}}$, which correspond to the non zero components of vector $s(X)$.

For the purpose of further consideration an operator $\otimes$ will be introduced, to multiplication of zero-one matrices. The operator $\otimes$ operates similarly to the multiplication of numerical matrix operator, except that the summation operation occurring in the multiplication process carries out the operator $\oplus$, which operating rules are defined as follows:

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

Let $A = [a_{ij}]_{I \times J}$ and $B = [b_{li}]_{L \times I}$ denote zero-one matrices, such that

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1J} \\ a_{21} & a_{21} & \dots & a_{2J} \\ & & \dots & \\ a_{I1} & a_{I2} & \dots & a_{IJ} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1L} \\ b_{21} & b_{21} & \dots & a_{2L} \\ & & \dots & \\ b_{J1} & b_{J2} & \dots & b_{JL} \end{bmatrix}$$

wherein $a_{ij}, b_{li} \in \{0,1\}$, $i = 1,2,...I$, $j = 1,2,...J$, $l = 1,2,...L$. Then

$$A \otimes B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1J} \\ a_{21} & a_{21} & \dots & a_{2J} \\ & & \dots & \\ a_{I1} & a_{I2} & \dots & a_{IJ} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1L} \\ b_{21} & b_{21} & \dots & a_{2L} \\ & & \dots & \\ b_{J1} & b_{J2} & \dots & b_{JL} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1L} \\ c_{21} & c_{21} & \dots & c_{2L} \\ & & \dots & \\ c_{I1} & c_{I2} & \dots & c_{IL} \end{bmatrix},$$

where $c_{il} \in \{0,1\}$, $c_{il} = \sum_{n=1}^{I} a_{in} b_{nl}$ and the operator $\sum$ performs the summation using a zero-one summation operator $\oplus$.

Example

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

For a given set of stops $\boldsymbol{P}$ and the matrix $X \in \boldsymbol{X}$ to define assignment students to particular bus stops, the vector $s( X )$ can be calculated as the logical product of a matrix $X$ and $\overline{P}$ - component column vector $\boldsymbol{1}$, consisting of the same "ones", i.e.

$$s( X ) = [X \otimes \boldsymbol{1}]^T , \tag{4}$$

where the symbol $T$ means transposed column vector obtained as a result of the multiplication $X \otimes \boldsymbol{1}$. The number of active stations $L(X)$ can be determined as follows:

$$L( X ) = \sum_{\overline{p}=1}^{\overline{P}} s_{\overline{p}}( X ) , \tag{5}$$

where $s_{\overline{p}}( X )$ is the $\overline{p} - th$ component of the vector $s( X )$ defined by equation (3).

Example

$$\text{If } X = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ then:}$$

$$s(X)=\begin{bmatrix}0&1&1&0&1&0&0&1&1&0&1\\1&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&1&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0\end{bmatrix}\otimes\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\\0\end{bmatrix},$$

where, after the transposition of the resulting column vector, we have $s(X)=[1,1,1,1,0]$, i.e. $\boldsymbol{P}=\{1,2,3,4\}\subset\overline{\boldsymbol{P}}=\{1,2,3,4,5\}$. The number of active bus stops $L(X)=\sum_{\overline{p}=1}^{5}s_{\overline{p}}(X)=4$.

Based on the assumptions and symbols the problem of defining a minimum set of school bus stops and their assignments to authorized to transport students may be formulated as the following one criterion optimization problem:
having:

- a set of numbers of potential bus stops $\overline{\boldsymbol{P}}=\{1,2,3,...,\overline{p},...,\overline{P}\}$;
- set of the numbers of all authorized to transport students $\boldsymbol{U}=\{1,2,3,...,u,...,U\}$;
- the maximum allowable distance from the place of residence of the student's and bus stops $d_{max}$

find a matrix $X=[x_{\overline{p}u}]_{\overline{P}\times U}$ that minimize a function

$$L(X)=\sum_{\overline{p}=1}^{\overline{P}}s_{\overline{p}}(X),\ X\in\boldsymbol{X}, \tag{6}$$

where the set of feasible solutions $X$ determine, in accordance with previous observations, the following constraints:

- $X=[x_{\overline{p}u}]_{\overline{P}\times U}$ wherein the elements $x_{\overline{p}u}$ satisfy the condition (2) (7)
- $\sum_{u=1}^{U}x_{\overline{p}u}\geq 0,\quad p\in\overline{\boldsymbol{P}},\ u\in\boldsymbol{U},$ (8)

- $$\sum_{\overline{p}=1}^{\overline{P}} x_{\overline{p}u} = 1,\ p \in \overline{\boldsymbol{P}},\ u \in \boldsymbol{U}\ . \qquad (9)$$

## 4. THE METHOD OF SOLVING THE PROBLEM OF DETERMINING THE MINIMUM SET OF SCHOOL BUS STOPS

In the following algorithm to determine the solution of the optimization problem (6) - (10) will be used the operation $p \quad q$ of the reduction of the $p$-th row of the matrix $X$ to the $q$-th row of this matrix, $p, q \in \overline{\boldsymbol{P}}$, defined as follows:

| ***p*** | ***q*** | ***p  q*** |
|---|---|---|
| *0* | *0* | *0* |
| *0* | *1* | *0* |
| *1* | *0* | *1* |
| *1* | *1* | *0* |

<u>Example</u>

Let

$$X = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then, after reduction of the remaining rows of the matrix $X$ to the row $q = 3$ we obtain the matrix

$$X' = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

in turn, the reduction of its rows to row $q = 4$ gives a matrix

$$X''=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

To solve the problem (6) - (9) the following algorithm can be used:

1. Create a matrix $\overline{X}=[\overline{x}_{\overline{p}u}]_{\overline{P}\times U}$, $\overline{p}\in\overline{\boldsymbol{P}}$, $u\in\boldsymbol{U}$. Set $X=\overline{X}$.
2. If for each $u\in\boldsymbol{U}$ occurs $\sum_{\overline{p}=1}^{\overline{P}} x_{\overline{p}u}=1$ then $s(X)=[X\otimes\boldsymbol{1}]^T$. Stop.
3. Create a set $\boldsymbol{J}\subseteq\overline{\boldsymbol{P}}$ such that $j\in\boldsymbol{J}$ if there is $u\in\boldsymbol{U}$ such that $\sum_{\overline{p}=1}^{\overline{P}} x_{\overline{p}u}=1$, and $\overline{x}_{ju}=1$. Set $\boldsymbol{P}=\boldsymbol{J}$.
4. If the set $\boldsymbol{J}$ is not empty, go to step 5; otherwise go to step 11.
5. In the set $\boldsymbol{J}$ designate an element for which the sum $\sum_{u=1}^{U} x_{nu}$ is the largest; if there are more than one such elements, take any of them; make a reduction of the remaining rows of the matrix $X$ to the row designated by this element; remove its number item from the set $\boldsymbol{J}$. If for each $u\in\boldsymbol{U}$ occurs $\sum_{\overline{p}=1}^{\overline{P}} x_{\overline{p}u}=1$ then $s(X)=[X\otimes\boldsymbol{1}]^T$. Stop.
6. If the set $\boldsymbol{J}$ is not empty, go to step 5; otherwise skip to step 7.
7. In the set $\boldsymbol{U}\setminus\boldsymbol{P}$ find element for which the sum $\sum_{u=1}^{U} x_{nu}$ is the largest; if there are more than one such elements, take any of them; make a reduction of the remaining rows of the matrix $X$ to the row designated by this element; add this element to the set $\boldsymbol{P}$.
8. If for each $u\in\boldsymbol{U}$ there is $\sum_{\overline{p}=1}^{\overline{P}} x_{\overline{p}u}=1$ then $s(X)=[X\otimes\boldsymbol{1}]^T$. Stop.
9. If the set $\boldsymbol{U}\setminus\boldsymbol{P}$ is not empty, go to step 7; otherwise $s(X)=[X\otimes\boldsymbol{1}]^T$. Stop.

The presented algorithm to solve the problem of determining the minimum set of school bus stops is a greedy algorithm, because at every stage of the transformation of matrix *X* is selected row containing the largest number of "ones".

## 5. NUMERICAL EXAMPLE

To illustrate functioning of the proposed algorithm to solve the problem of determining the minimum set of school bus stops a simple numerical example will be presented.
Data:

- a set of numbers of potential bus stops $\overline{\boldsymbol{P}} = \{1,2,3,4,5\}$;
- a set the numbers of all students authorized to transport $\boldsymbol{U} = \{1,2,3,4,5,6,7,8,9,10,11\}$;
- the maximum allowable distance between the place of residence of the students and particular bus stops dmax = 15 minutes;
- distance matrix $D = [d_{\overline{p}u}]_{\overline{P} \times U}$ of individual places of student residence, authorized to transport from particular potential bus stops:

$$D = \begin{bmatrix} 20 & 8 & 10 & 25 & 5 & 30 & 45 & 15 & 12 & 20 & 10 \\ 15 & 30 & 25 & 12 & 65 & 20 & 35 & 40 & 70 & 28 & 45 \\ 25 & 20 & 40 & 60 & 10 & 15 & 30 & 8 & 20 & 10 & 28 \\ 10 & 14 & 30 & 25 & 30 & 45 & 15 & 15 & 35 & 25 & 60 \\ 50 & 40 & 10 & 5 & 25 & 25 & 35 & 60 & 15 & 70 & 10 \end{bmatrix}.$$

According to (1) matrix $\overline{X} = [\overline{x}_{\overline{p}u}]_{\overline{P} \times U}$ that determines the potential assignment of students to potential school bus stops is:

$$\overline{X} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Thus, after step 1 of the algorithm, matrix *X* in accordance with (2) has the form

$$X = \overline{X} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Because the matrix $X$ has the columns containing several "ones" we go to step 3 of the algorithm. The set $\boldsymbol{J}$ has the form $\boldsymbol{J} = \{3,4\}$, similarly as the set $\boldsymbol{P} = \{3,4\}$. Since the set $\boldsymbol{J}$ is not empty, we go to step 5 of the algorithm. Since the sum of "ones" in the number of rows *3* and *4* are the same, we do reduction of the remaining rows of the matrix $X$ to the first of these rows, i.e. row number 3 obtaining the matrix

$$X = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

According to step 5 of the algorithm, after the reduction of the remaining rows of the matrix $X$ to the row number 3 we remove this row number from the set $\boldsymbol{J}$. After this operation $\boldsymbol{J} = \{4\}$. Because the resulting matrix $X$ contains columns with multiple "ones" we go to step 6 of the algorithm, according to which the reduction is performed with respect to the row 4 and we obtain the matrix

$$X = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

After that, a set $\boldsymbol{J}$ is empty, i.e. $\boldsymbol{J} = \{\}$. In accordance with step 7 of the algorithm in the set $\boldsymbol{U} \setminus \boldsymbol{P} = \{1,2,5\}$ we are setting the row with the largest number of "ones". This is the row number *5*. After the reduction of the remaining rows of the matrix $X$ to the row number 5 we get:

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

After performing this operation $\boldsymbol{P} = \{3,4,5\}$. After we go to step 8 of the algorithm, we find that for each $u \in \boldsymbol{U} = \{1,2,3,4,5,6,7,8,9,10,11\}$ there is $\sum_{\bar{p}=1}^{5} x_{\bar{p}u} = 1$. Therefore, the vector of active school bus stops $s(X) = (s_1(X), s_2(X),. s_3(X), s_4(X), s_5(X))$, according to the equation (4) has the form

$$s(X) = [X \otimes \boldsymbol{1}]^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the set of school bus stops with a minimum cardinality has the form $\boldsymbol{P} = \{3,4,5\}$ and the number of stops is $L(X) = \sum_{\bar{p}=1}^{\bar{P}} s_{\bar{p}}(X) = \sum_{\bar{p}=1}^{5} s_{\bar{p}}(X) = 3$.

## 6. CONCLUSION

In this paper a formal presentation and description of a method of solving the problem of both determining the set of bus stops and assignment of students that are authorized to transport to these stops is investigated. This problem can be treated as a subproblem of the bus school routing problem (SBRP). Although the problems of the SBRP class are one of the earliest logistics problems solved using methods of operations research, they remain valid and are the subject of research, as evidenced by numerous contemporary publications. It is worth noting that in most of the problems of SBRP class described in the literature the problem of determining the bus stops network and allocation of students to the particular stops is very often ignored (it is assumed that the bus stops location and assignment of students to them are given). Based on the assumption that a small number of bus stops, from which the students are taken or to which they are set down, makes carrying out of school transport process easier, a problem of minimizing the number of active bus stops was considered. The main result of this paper is proposition of a greedy algorithm to solving the problem of determining the minimum set of school bus stops is. To illustrate functioning of the proposed algorithm a simple numerical example has been presented.

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## BIOGRAPHICAL NOTES

**Kazimierz Worwa** is an Associate Professor at the Cybernetics Faculty of Military Technical University in Warsaw. His research interests include software reliability modelling, software testing methods and software efficiency. He is the author and co-author of many scientific publications. His paper appear in numerous journals, e.g. Control and Cybernetics or Polish Journal of Environmental Studies and proceedings of international conferences, eg. Depcos-Relcomex, Information Management or System, Modelling and Control.